

**FUNDAMENTAL WEIGHTS, PERMUTATION WEIGHTS
AND WEYL CHARACTER FORMULA**

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Abstract

For a finite Lie algebra G_N of rank N , the Weyl orbits $W(\Lambda^{++})$ of strictly dominant weights Λ^{++} contain $\dim W(G_N)$ number of weights where $\dim W(G_N)$ is the dimension of its Weyl group $W(G_N)$. For any $W(\Lambda^{++})$, there is a very peculiar subset $\wp(\Lambda^{++})$ for which we always have

$$\dim \wp(\Lambda^{++}) = \dim W(G_N) / \dim W(A_{N-1}) .$$

For any dominant weight Λ^+ , the elements of $\wp(\Lambda^+)$ are called **Permutation Weights**.

It is shown that there is a one-to-one correspondence between elements of $\wp(\Lambda^{++})$ and $\wp(\rho)$ where ρ is the Weyl vector of G_N . The concept of signature factor which enters in Weyl character formula can be relaxed in such a way that signatures are preserved under this one-to-one correspondence in the sense that corresponding permutation weights have the same signature. Once the permutation weights and their signatures are specified for a dominant Λ^+ , calculation of the character $ChR(\Lambda^+)$ for irreducible representation $R(\Lambda^+)$ will then be provided by A_N multiplicity rules governing generalized Schur functions. The main idea is again to express everything in terms of the so-called **Fundamental Weights** with which we obtain a quite relevant specialization in applications of Weyl character formula. To provide simplifications in practical calculations, a reduction formula governing classical Schur functions is also given. As the most suitable one, the E_6 example which requires a sum over 51840 Weyl group elements, is studied explicitly. This will be instructive also for an explicit application of A_5 multiplicity rules.

In result, it will be seen that Weyl or Weyl-Kac character formulas find explicit applications no matter how big is the rank of underlying algebra.

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I. INTRODUCTION

It is well-known that summations over Weyl groups of Lie algebras enter in many areas of physics as well as in mathematics. They are at the heart of all character calculations for finite [1] and also affine [2] Lie algebras and hence are of great importance in calculations of weight multiplicities [3] or in decompositions [4] of tensor products of irreducible representations. In high energy physics, it is known that calculations of fusion coefficients [5] or S-matrices which appear in modular transformations [6] of affine characters are directly related to summations over Weyl groups. This however is not an easy task except for a few cases which correspond to some Lie algebras of low rank. Let us emphasize, for instance, that the summations are over respectively 51840, 2903040 and 696729600 Weyl group elements for E_6, E_7 and E_8 Lie algebras. It is therefore worthwhile to study the problem more closely.

In a previous work [7] we have shown that in applications of Weyl character formula for A_N Lie algebras the sums over Weyl groups can be represented by permutations. This in essence is being in line with the fact that A_N Weyl groups are already the permutation groups of $(N+1)$ objects. It is however interesting to note that this can be seen only when one uses some properly chosen set of weights which we call **fundamental weights**. We have also shown that the signatures of Weyl reflections can then be given precisely as being in relations with these permutations. One could therefore expect that there is a way to extend this procedure to any other finite Lie algebra \mathbf{G}_N in view of the fact that it always has an A_{N-1} sub-algebra.

For this, let us recall that, one has, for any dominant weight Λ^+ of \mathbf{G}_N , a strictly dominant weight $\Lambda^{++} \equiv \Lambda^+ + \rho$ where ρ is the Weyl vector of G_N . The character $ChR(\Lambda^+)$ of the corresponding irreducible representation $R(\Lambda^+)$ is then given by

$$ChR(\Lambda^+) = \frac{A(\Lambda^{++})}{A(\rho)} \quad (I.1)$$

where

$$A(\mu) \equiv \sum_{\omega} \epsilon(\omega) e^{\omega(\mu)} \quad (I.2)$$

can be defined for any weight μ . The sum here is over Weyl group $W(G_N)$ and $\epsilon(\omega)$ is the signature of the Weyl reflection ω . The main emphasis now is on the fact that, for any strictly positive dominant weight Λ^{++} , the number of elements of Weyl orbit $W(\Lambda^{++})$ is always equal to the dimension of the corresponding Weyl group. This hence allows us to re-write (I.2) in the form

$$A(\Lambda^{++}) \equiv \sum_{\mu \in W(\Lambda^{++})} \epsilon(\mu) e^{\mu} \quad (I.3)$$

where $W(\Lambda^{++})$ is the corresponding Weyl orbit. One must immediately note here that the concept of signature encountered in (I.2) is conveniently relaxed in (I.3) in such a way that we introduce a signature $\epsilon(\mu)$ for each and every weight μ within the Weyl orbit $W(\Lambda^{++})$. It will be seen in the following that (I.3) is a quite relevant form of (I.2) if one aims to apply it in Weyl character formula (I.1). To this end, the concept of **Permutation Weight** is of central importance.

II. PERMUTATION WEIGHTS

It is known that the branching rules $\mathbf{G}_N \rightarrow A_{N-1}$ give us irreducible A_{N-1} representations which participate in the decomposition of an irreducible representation of G_N . We, instead, want here to make the same for Weyl orbits rather than representations. For this, the following definition seems to be useful:

A Weyl orbit $W(\Lambda^+)$ always includes a sub-set $\wp(\Lambda^+)$ of weights having the form

$$\sum_{i=1}^{N-1} k_i \lambda_i - k \lambda_N \quad , \quad k_i \in \mathbb{Z}^+ \quad , \quad k \in \mathbb{Z} \quad (II.1)$$

where $\mathbb{Z}(\mathbb{Z}^+)$ is the set of integers (positive integers). The elements of $\wp(\Lambda^+)$ are called **permutation weights** of Λ^+ .

λ_I 's and α_I 's ($I=1,2,\dots,N$) are respectively the fundamental dominant weights and the simple roots of G_N . For details of Lie algebra technology we refer to the excellent book of Humphreys [8]. As will be seen from the permutational lemma given in our previous works [9], Weyl orbits of A_N Lie algebras are stable under permutations and this hence allows us to determine the complete weight structure of an A_N Weyl orbit. The permutation weights will give us the same possibility but for any finite Lie algebra G_N other than A_N Lie algebras. We will therefore show now an explicit way to obtain all permutation weights of a Weyl orbit $W(\Lambda^+)$ of G_N .

Let us first emphasize by definition that the sum of two permutation weights is again a permutation weight. Let $\wp(\lambda)$ and $\wp(\lambda')$ be the sets of permutation weights for λ and λ' . It is then clear that

$$\wp(\lambda + \lambda') \subset \wp(\lambda) \cup \wp(\lambda') \quad (II.2)$$

and for any element $\mu \in \wp(\lambda) \cup \wp(\lambda')$ one can also state $\mu \in \wp(\lambda + \lambda')$ on condition that

$$(\mu, \mu) = (\lambda + \lambda', \lambda + \lambda') \quad (II.3)$$

where $(.,.)$ is the scalar product which can be introduced on the weight lattice of \mathbf{G}_N . It is therefore sufficient to know $\wp(\lambda_I)$'s ($I=1,2,\dots,N$) in order to obtain the set $\wp(\Lambda^+)$ of permutation weights for any dominant weight Λ^+ which is known to be expressed by

$$\Lambda^+ = \sum_{I=1}^N k_I \lambda_I, \quad k_I \in \mathbb{Z}^+.$$

We find convenient here to exemplify our work in the Lie algebra of E_6 with the following Coxeter-Dynkin diagram:

$$\begin{array}{ccccc} & & 6 & & \\ 1 & 2 & 3 & 4 & 5 \end{array}$$

The permutation weight subsets of its fundamental Weyl orbits will then be given by

$$\begin{aligned} \wp(\lambda_1) &\equiv \{\lambda_1, \lambda_1 - \lambda_6, \lambda_4 - \lambda_6\}, \\ \wp(\lambda_2) &\equiv \{\lambda_2, \lambda_2 - 2\lambda_6, \lambda_3 + \lambda_5 - 2\lambda_6, \lambda_1 + \lambda_4 - \lambda_6, \lambda_1 + \lambda_4 - 2\lambda_6, 2\lambda_1 - \lambda_6\}, \\ \wp(\lambda_3) &\equiv \{\lambda_3, \lambda_1 + \lambda_3 + \lambda_5 - 3\lambda_6, \lambda_1 + \lambda_3 + \lambda_5 - 2\lambda_6, \lambda_2 + 2\lambda_5 - 2\lambda_6, \\ &\quad \lambda_2 + \lambda_4 - 3\lambda_6, \lambda_2 + \lambda_4 - \lambda_6, \lambda_3 - 3\lambda_6, 2\lambda_3 - 3\lambda_6, 2\lambda_1 + \lambda_4 - 2\lambda_6\}, \\ \wp(\lambda_4) &\equiv \{\lambda_4, \lambda_4 - 2\lambda_6, \lambda_1 + \lambda_3 - 2\lambda_6, \lambda_2 + \lambda_5 - \lambda_6, \lambda_2 + \lambda_5 - 2\lambda_6, 2\lambda_5 - \lambda_6\}, \\ \wp(\lambda_5) &\equiv \{\lambda_5, \lambda_2 - \lambda_6, \lambda_5 - \lambda_6\}, \\ \wp(\lambda_6) &\equiv \{\lambda_6, -\lambda_6, \lambda_1 + \lambda_5 - \lambda_6, \lambda_3 - \lambda_6, \lambda_3 - 2\lambda_6\}. \end{aligned} \quad (II.4)$$

In the notation of $(k_1, k_2, k_3, k_4, k_5, k_6)$ for $\sum_{I=1}^6 k_I \lambda_I$, half of the 72 elements of $\wp(\rho)$ can now be chosen, by direct use of (II.3), among elements of $\sum_{I=1}^6 \wp(\lambda_I)$ as in the following:

$$\begin{aligned} \rho(1) &= (1, 1, 1, 1, 1, 1), \quad \rho(13) = (3, 2, 2, 1, 2, -6), \quad \rho(25) = (3, 1, 3, 2, 1, -8) \\ \rho(2) &= (1, 1, 2, 1, 1, -1), \quad \rho(14) = (2, 1, 2, 2, 3, -6), \quad \rho(26) = (6, 1, 1, 2, 1, -7) \\ \rho(3) &= (1, 2, 1, 2, 1, -2), \quad \rho(15) = (1, 3, 1, 3, 1, -6), \quad \rho(27) = (1, 2, 1, 1, 6, -7) \\ \rho(4) &= (1, 3, 1, 1, 2, -3), \quad \rho(16) = (4, 2, 1, 1, 3, -6), \quad \rho(28) = (2, 2, 2, 1, 4, -8) \\ \rho(5) &= (2, 1, 1, 3, 1, -3), \quad \rho(17) = (3, 1, 1, 2, 4, -6), \quad \rho(29) = (4, 1, 2, 2, 2, -8) \\ \rho(6) &= (2, 2, 1, 2, 2, -4), \quad \rho(18) = (4, 1, 3, 1, 1, -7), \quad \rho(30) = (2, 1, 4, 1, 2, -9) \\ \rho(7) &= (1, 4, 1, 1, 1, -4), \quad \rho(19) = (1, 1, 3, 1, 4, -7), \quad \rho(31) = (7, 1, 1, 1, 1, -7) \\ \rho(8) &= (1, 1, 1, 4, 1, -4), \quad \rho(20) = (2, 2, 2, 2, 2, -7), \quad \rho(32) = (1, 1, 1, 1, 7, -7) \\ \rho(9) &= (3, 1, 2, 1, 3, -5), \quad \rho(21) = (5, 1, 2, 1, 2, -7), \quad \rho(33) = (1, 3, 1, 1, 5, -8) \\ \rho(10) &= (2, 3, 1, 2, 1, -5), \quad \rho(22) = (2, 1, 2, 1, 5, -7), \quad \rho(34) = (5, 1, 1, 3, 1, -8) \\ \rho(11) &= (1, 2, 1, 3, 2, -5), \quad \rho(23) = (3, 2, 1, 2, 3, -7), \quad \rho(35) = (3, 1, 3, 1, 3, -9) \\ \rho(12) &= (4, 1, 1, 1, 4, -5), \quad \rho(24) = (1, 2, 3, 1, 3, -8), \quad \rho(36) = (1, 1, 5, 1, 1, -10) \end{aligned} \quad (II.5)$$

III. EXPLICIT CONSTRUCTION OF WEYL ORBITS

It is known that the complete set of weights of a Weyl orbit is obtained by the fact that Weyl orbits are by definition stable under Weyl reflections. Instead, we want to construct Weyl orbits here by knowing their permutation weights solely. As in above, let λ_I 's be the fundamental dominant weights of \mathbf{G}_N whereas σ_i 's be the ones for its A_{N-1} sub-algebra.

The existence of such a sub-algebra can always be shown explicitly by taking

$$\sigma_i = \lambda_i - n_i \lambda_N \quad (III.1)$$

where n_i 's are some specified rational numbers. Let us recall from our previous references [7, 9] that fundamental weights μ_I ($I=1,2,\dots,N$) for A_{N-1} sub-algebra are defined by

$$\sigma_i \equiv \mu_1 + \mu_2 + \dots + \mu_i \quad , \quad i = 1, 2, \dots, N-1. \quad (III.2)$$

together with the condition that

$$\mu_1 + \mu_2 + \dots + \mu_N \equiv 0 \quad (III.3)$$

and also

$$(\mu_I, \lambda_N) \equiv 0 \quad . \quad (III.4)$$

The permutational lemma then states for an A_{N-1} dominant weight

$$\sigma^+ = s_1 \mu_1 + s_2 \mu_2 + \dots + s_N \mu_N \quad , \quad s_1 \geq s_2 \geq \dots \geq s_N \geq 0 \quad (III.5)$$

that its Weyl orbit $W(\sigma^+)$ are obtained to be

$$W(\sigma^+) = \{s_1 \mu_{I_1} + s_2 \mu_{I_2} + \dots + s_N \mu_{I_N}\} \quad (III.6)$$

by permutating fundamental weights μ_I 's. Note here that no two of indices I_1, I_2, \dots, I_N ($=1,2,\dots,N$) shall take the same value. This is also true for all permutation weights because, for $\lambda_N \rightarrow 0$, they turn out to be A_{N-1} dominant weights. We then obtain an extension of the permutational lemma for any finite Lie algebra other than A_N Lie algebras.

An example will again be helpful here. Let us consider $E_6 \rightarrow A_5$ decomposition which is specified by

$$\sigma_1 = \lambda_1 - \frac{1}{2} \lambda_6 \quad , \quad \sigma_2 = \lambda_2 - \frac{2}{2} \lambda_6 \quad , \quad \sigma_3 = \lambda_3 - \frac{3}{2} \lambda_6 \quad , \quad \sigma_4 = \lambda_4 - \frac{2}{2} \lambda_6 \quad , \quad \sigma_5 = \lambda_5 - \frac{1}{2} \lambda_6 \quad (III.7)$$

where λ_I 's ($I=1,2,\dots,6$) are E_6 fundamental dominant weights while σ_i 's ($i=1,2,\dots,5$) are those of A_5 . The influence of A_5 permutational lemma for E_6 Weyl orbits can be illustrated, in view of (II.4), in the following example:

$$W(\lambda_1) = \{W(\sigma_1) + \frac{1}{2} \Omega \quad , \quad W(\sigma_1) - \frac{1}{2} \Omega \quad , \quad W(\sigma_4)\} \quad (III.9)$$

where, for A_5 Weyl orbits, we know that

$$W(\sigma_1) = \{\mu_{I_1}\} \quad , \quad W(\sigma_4) = \{\mu_{I_1} + \mu_{I_2} + \mu_{I_3} + \mu_{I_4}\} \quad , \quad I_1 \geq I_2 \geq I_3 \geq I_4 = 1, 2, \dots, 6 \quad .$$

In (III.9), one keeps the notation $\Omega \equiv \lambda_6$ for which we know that $(\Omega, \mu_I) = 0$. It is, in fact, nothing but an example of the **branching rule of Weyl orbits** which is at the heart of our definition of permutation weights. The branching rules for the remaining fundamental E_6 Weyl orbits $W(\lambda_i)$ for $i=2,3,\dots,6$ can be obtained similarly from the permutation weights given in (II.4).

What we want to emphasize here is mainly that 72 permutation weights of $W(\rho)$ of E_6 will be given by

$$\wp(\rho) = (\sigma(k)^{++} + r(k) \Omega \quad , \quad \sigma(k)^{++} - r(k) \Omega) \quad (III.10)$$

where the Ω extension parameters $r(k)$'s are some **positive** rational numbers. 36 strictly dominant weights $\sigma(k)^{++}$ and their parameters $r(k)$'s can be determined from (II.5) respectively ($k=1,2,\dots,36$).

IV. APPLICATIONS OF WEYL FORMULA AND A LEMMA

Since we have in mind to make summations over Weyl groups explicitly, the decomposition (III.10) gives us for E_6 the possibility to calculate $A(\rho)$ by the aid of (I.3). Let us recall from ref.[7] that for anyone of the A_5 dominant weights

$$\sigma^{++}(k) \equiv \sigma^+(k) + \rho_\sigma$$

participated in the list (III.10), one has

$$A(\sigma^{++}(k) + r(k) \Omega) = A(\rho_\sigma) S(\sigma^+(k)) u^{r(k)} \quad (IV.1)$$

where ρ_σ is the Weyl vector of A_5 . In the specialization

$$e^\Omega \equiv u, e^{\mu_I} \equiv u_I, I = 1, 2, \dots, 6 \quad (IV.2)$$

of formal exponentials, we know that $S(\sigma^+(k))$ is a generalized Schur function [7, 10] which can be reduced via A_5 multiplicity rules to a polynomial expression in terms of 5 indeterminates x_i ($i=1, 2, \dots, 5$) which are defined by

$$u_1^M + u_2^M + u_3^M + u_4^M + u_5^M + u_6^M \equiv M x_M, M = 1, 2, \dots \quad (IV.3)$$

Note here as a result of (III.3) that 6 indeterminates u_I are constrained by

$$\prod_{I=1}^6 u_I \equiv 1 \quad (IV.4)$$

and hence one can immediately see from definitions (IV.3) that, for $M > 5$, all indeterminates x_M depend non-linearly on the first five indeterminates x_i ($i=1, 2, \dots, 5$). This will also give rise to some reduction rules governing classical Schur functions [7] which are defined by

$$S(M \lambda_1) \equiv S_M(x_1, x_2, \dots, x_5), M = 1, 2, \dots, 5, 6, \dots \quad (IV.5)$$

where $S_M(x_1, x_2, \dots, x_5)$'s are some polynomials which can be obtained for $M=1, 2, \dots, 5$ directly. For $M > 5$, however, one must take into account the above mentioned non-linear relations among indeterminates x_M . Practical calculations could get complicated in general for A_N multiplicity rules. For this, we find convenient to give here some clarifying details. It will be seen in fact that these non-linear relations governing indeterminates x_M for $M > N$, result in the following reduction rules among polynomials $S_M(x_1, x_2, \dots, x_N) \equiv S_M(N)$ which correspond to classical Schur functions as in (IV.5):

$$S_M(N) = (-1)^N S_{M-N-1}(N) - \sum_{i=1}^N S_i^*(N) S_{M-i}(N), M > N \quad (IV.6)$$

where $S_M^*(N)$ is obtained from $S_M(N)$ under the replacements $x_i \rightarrow -x_i$. It will be seen that the reduction rules given in (IV.6) prove extremely useful in applications of A_N multiplicity rules especially for higher values of the rank N .

Another important notice here is to give a precise definition of signatures for 72 permutation weights participated in the decomposition (III.10). The arrangement in (III.10) is in such a way that

$$\begin{aligned} \epsilon(\sigma^{++}(k) + r(k) \Omega) &\equiv +1 \\ \epsilon(\sigma^{++}(k) - r(k) \Omega) &\equiv -1 \end{aligned} \quad (IV.7)$$

for $k=1, 2, \dots, 36$. The miraculous factorization (IV.1) of the Weyl formula comes out only by the aid of such a choice.

It is thus seen that the decomposition (III.10) of $\wp(\rho)$ allows us to calculate $A(\rho)$ but nothing says about any other $A(\Lambda^{++})$ which we need in the calculation of the character $ChR(\Lambda^+)$. For this, a lemma which assures one-to-one correspondence between 72 elements of (III.10) and those of any other $\wp(\Lambda^{++})$ would be

of great help. In view of the condition (II.3), there is a one-to-one correspondence which maps any element of $\wp(\rho)$ to one and only one element of $\wp(\Lambda^{++})$ in such a way that their signatures are preserved. The generalization leads us to the following **lemma**:

Let, for any dominant weight Λ^+ , $\wp(\Lambda^+)$ be the subset of its permutation weights and also

$$\wp(\rho) \equiv \{\rho(k)\} , \quad \wp(\Lambda^{++}) \equiv \{\Lambda(k)\} \quad (IV.8)$$

for any Lie algebra G_N with the Weyl vector ρ . Then, in view of condition (II.3), for

$$k = 1, 2, \dots \dim W(G_N) / \dim W(A_{N-1})$$

and $\mu \in \wp(\Lambda^+)$ there is a one-to-one correspondence Ξ which provides

$$\Xi : \rho(k) + \mu \rightarrow \Lambda(k) \quad (IV.9)$$

in such a way that

$$\epsilon(\Xi(\rho(k))) \equiv \epsilon(\rho(k)) . \quad (IV.10)$$

Note here that, we always have

$$\dim \wp(\Lambda^{++}) \geq \dim \wp(\Lambda^+)$$

and for each and every value of k there is one and only one $\mu \in \wp(\Lambda^+)$.

In the conclusion, we can say that the decomposition (II.5) makes any explicit summation over 51840 elements of E_6 Weyl group possible and hence completely solves the problem for E_6 Lie algebra. One must however add that related definitions must be made precisely case by case for any other Lie algebra. For all the chains B_N, C_N, D_N , the exceptional Lie algebras G_2, F_4 and even for E_7 the method presented above is easily tractable as we will show in a subsequent paper. The same could also be true for E_8 but again one must note that we have $\dim \wp(\rho) = 17280$ for E_8 . We finally remark that a similar analysis can be presented in the framework of A_8 sub-algebra of E_8 and this makes the problem more tractable by reducing the number of permutation weights down to 1920. To the knowledge of authors, this is quite convenient to handle any problem which requires summations over 696729600 elements of E_8 Weyl group in an explicit manner and hence it would be worthwhile to study in another publication.

As the last but not the least, let us add that all these calculations can be performed by the aid of very simple computer programs, say, in the language of Mathematica [11].

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